On the total irregularity strength of wheel related graphs

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Abstract

A totally irregular total k-labeling $f : V \cup E \to \{1, 2, 3, \ldots, k\}$ is a labeling of vertices and edges of G in such a way that for any two different vertices x and y their vertex-weights $wt_h(x) \neq wt_h(y)$ where the vertex-weight $wt_h(x) = h(x) + \sum_{xy \in E} h(xz)$ and also for every two different edges xy and $x'y'$ of G their edge-weights $wt_h(xy) = h(x) + h(xy) + h(y)$ and $wt_h(x'y') = h(x') + h(x'y') + h(y')$ are distinct. A total irregularity strength of graph G, denoted by $ts(G)$ is defined as the minimum k for which a graph G has a totally irregular total k-labeling. In this paper, we investigate some wheel related graphs whose *total irregularity strength* equals to the lower bound.

Keywords: vertex irregular total k-labeling; edge irregular total k-labeling; total irregularity strength.
AMS Classification (2010): 05C78

1 Introduction

Let G be a finite, simple, and undirected graph with the vertex set V and edge set E. A labeling of a graph G is a mapping that carries a set of graph elements into a set of numbers (usually to positive or non-negative integer). If the domain of mapping is a vertex set, or an edge set or a union of vertex and edge set, then the labeling is called *vertex labeling , edge labeling or total labeling* , respectively. Bača et.al [3] defined a labeling $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, k\}$ to be a vertex irregular total k-labeling if for every two different vertices x and y the vertex-weights $wt_f(x) \neq wt_f(y)$ where the vertex-weight $wt_f(x) = f(x) + \sum_{xz \in E} f(xz)$. A minimum k for which G has a vertex irregular total k-labeling is called as the

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total vertex irregularity strength of G and denoted by $tvs(G)$. They obtained the exact values of the total vertex irregularity strength of cycle, star, complete graph and prism. Moreover, Nurdin et al. [11] proved the exact value of the total vertex irregularity strength for any tree T with n pendant vertices and no vertex of degree two, that is $tvs(T) = \lceil \frac{n+1}{2} \rceil$. Bača et al. [3] defined a labeling $\phi : V \cup E \rightarrow \{1, 2, ..., k\}$ to be an edge irregular total k labeling of the graph G if for every two different edges xy and $x'y'$ of G the edge-weights $wt_\phi(uv) = \phi(u) + \phi(uv) + \phi(v)$ and $wt_\phi(u'v') = \phi(u') + \phi(u'v') + \phi(v')$ are different. The *total edge irregularity strength* $tes(G)$ is defined as the minimum k for which G has an edge irregular total k-labeling. Also, they obtained the exact values of the tes of path, cycle, star, wheel and friendship graph. Ivančo and Jendroľ [7] proved that

$$tes(G) \geq max\left\{\left\lceil \frac{(|E(G)| + 2)}{3} \right\rceil, \left\lceil \frac{(\Delta(G) + 1)}{2} \right\rceil\right\}. \quad (1)$$

Indriati et al. [4, 5] determined by the tes of generalized helm and generalized web graph. We found [8, 9] the total edge irregularity strength of the disjoint union of wheel graphs such as closed helm graph CH_n and flower graph Fl_n.

$$tes(CH_n) = \left\lceil \frac{4n + 2}{3} \right\rceil, n \geq 3. \quad (2)$$

$$tes(Fl_n) = \left\lceil \frac{4n + 2}{3} \right\rceil, n \geq 3. \quad (3)$$

Indriati et al. [6] found the total edge irregularity strength of the helm graph H_n.

$$tes(H_n) = n + 1, n \geq 3. \quad (4)$$

Ivančo and Jendroľ [7] posed the following conjecture.

Conjecture 1.1. [7] *Let G be an arbitrary graph different from K_5. Then*

$$tes(G) = max\left\{\left\lceil \frac{(|E(G)| + 2)}{3} \right\rceil, \left\lceil \frac{(\Delta(G) + 1)}{2} \right\rceil\right\}. \quad (5)$$

Moreover, they proved that for any tree T,

$$tes(T) = max\left\{\left\lceil \frac{(|E(T)| + 2)}{3} \right\rceil, \left\lceil \frac{(\Delta(T) + 1)}{2} \right\rceil\right\}. \quad (6)$$

Nurdin et al. [11] determined the lower bound of tvs for any graph G.

Theorem 1.2. *[11] Let G be a connected graph having n_i vertices of degree i ($i = \delta, \delta + 1, \delta + 2, ..., \Delta$) where δ and Δ are the minimum and maximum degree of*

G, respectively. Then

$$tvs(G) \geq max\left\{\left\lceil\frac{\delta+n_\delta}{\delta+1}\right\rceil, \left\lceil\frac{\delta+n_\delta+n_{\delta+1}}{\delta+2}\right\rceil, \ldots, \left\lceil\frac{\delta+\sum_{i=\delta}^{\Delta}(n_i)}{\Delta+1}\right\rceil\right\}. \quad (7)$$

Also Nurdin et al. [11] posed the following conjecture.

Conjecture 1.3. *[11] Let G be a connected graph having n_i vertices of degree i where δ and Δ are the minimum and maximum degree of G, respectively. Then*

$$tvs(G) = max\left\{\left\lceil\frac{\delta+n_\delta}{\delta+1}\right\rceil, \left\lceil\frac{\delta+n_\delta+n_{\delta+1}}{\delta+2}\right\rceil, \ldots, \left\lceil\frac{\delta+\sum_{i=\delta}^{\Delta}(n_i)}{\Delta+1}\right\rceil\right\}. \quad (8)$$

Ahmad et al. [1] found the total vertex irregularity strength of helm graph H_n and flower graph Fl_n.

$$tvs(H_n) = \left\lceil\frac{n+1}{4}\right\rceil, n \geq 4. \quad (9)$$

$$tvs(Fl_n) = \left\lceil\frac{2n+2}{5}\right\rceil, n \geq 4. \quad (10)$$

Combining the ideas of vertex irregular total k-labeling and edge irregular total k-labeling, Marzuki et al. [10] introduced another irregular total k-labeling called the *totally irregular total k-labeling*. A labeling $h : V(G) \cup E(G) \to \{1, 2, 3, \ldots, k\}$ to be a *totally irregular totalk-labeling* of the graph G if for every two different vertices x and y the vertex-weights $wt_h(x) \neq wt_h(y)$ where the vertex-weight $wt_h(x) = h(x) + \sum_{xz \in E} h(xz)$, and also for every two different edges xy and $x'y'$ of G the edge-weights $wt_h(xy) = h(x) + h(xy) + h(y)$ and $wt_h(x'y') = h(x') + h(x'y') + h(y')$ are distinct. The *total irregularity strength* $ts(G)$ is defined as the minimum k for which a graph G has a totally irregular total k-labeling. For the total irregularity strength of a graph G, they observed that

$$ts(G) \geq max\{tes(G), tvs(G)\}. \quad (11)$$

They also determined the total irregularity strength of cycles and paths. Ramdani and Salman [12] obtained the total irregularity strength of some Cartesian product graphs. Ramdani et al. [13] determined the total irregularity strength of gear graph $G_n, n \geq 3$, fungus graph $Fg_n, n \geq 3$ and disjoint union of star $mS_n, n, m \geq 2$. Also, Ali Ahmad et al. [2] obtained the total irregularity strength of generalized Petersen graph. In this paper, we investigate some wheel related graphs whose total irregularity strength equals to the lower bound. In addition, we show that these graphs admit totally irregular total k-labeling. Further we determine the exact value of their ts. We use the following definitions in the subsequent section.

Definition 1.4. *The helm graph H_n is obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle. Thus the vertex set of H_n is $V(H_n) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and the edge set of H_n is $E(H_n) = \{vv_i, v_iv_{i+1}, v_iu_i : 1 \leq i \leq n\}$ with indices taken modulo n.*

Definition 1.5. *The closed helm graph CH_n is obtained from a helm H_n by joining each pendant vertex to form a cycle. It contains three types of vertices: an apex of degree n, n vertices of degree 4 and n vertices of degree 3. Thus the vertex set of CH_n is $V(CH_n) = \{v_i, u_i, v : 1 \leq i \leq n\}$ and the edge set is $E(CH_n) = \{vv_i, v_iv_{i+1}, v_iu_i, u_iu_{i+1} : 1 \leq i \leq n\}$ with indices taken modulo n.*

Definition 1.6. *A double wheel graph DW_n of size 4n is composed of $2C_n + K_1$, that is it consists of two cycles of size n, where all the vertices of the two cycles are connected to a common hub. Thus the vertex set of DW_n is $V(DW_n) = \{v_i, u_i, v : 1 \leq i \leq n\}$ and the edge set is $E(DW_n) = \{vv_i, v_iv_{i+1}, vu_i, u_iu_{i+1} : 1 \leq i \leq n\}$ with indices taken modulo n.*

Definition 1.7. *The flower graph Fl_n is obtained from a helm by joining each pendant vertex to the central vertex of the helm. Thus the vertex set of Fl_n is $V(Fl_n) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and the edge set of Fl_n is $E(Fl_n) = \{vv_i, vu_i, v_iv_{i+1}, v_i u_i : 1 \leq i \leq n\}$, with indices taken modulo n.*

2 Main Results

In this section,we determine the total irregularity strength of helm graph H_n for $n \geq 3$, closed helm CH_n for $n \geq 3$, double wheel graph DW_n for $n \geq 3$ and flower graph Fl_n for $n \geq 3$.

Theorem 2.1. *Let $n \geq 3$ and H_n be a helm graph with 2n+1 vertices and 3n edges. Then $ts(H_n) = n + 1, n \geq 3$.*

Proof. Since $|V(H_n)| = 2n + 1$ and $|E(H_n)| = 3n$ by (4),(9) and (11) we have $ts(G) \geq n + 1$. For the reverse inequality, we define a total labeling f as follows : $f : V \cup E \rightarrow \{1, 2, 3, \ldots, n + 1\}$ by considering the following two cases.

Case(i): n is odd.

$f(u_iv_i) = f(v_i) = i, 1 \leq i \leq n;$
$f(u_i) = 1, 1 \leq i \leq n;$
$f(v) = n + 1;$
$f(vv_i) = n + 1, 1 \leq i \leq n;$

$f(v_iv_{i+1}) = 1, 1 \le i \le n-1;$
$f(v_nv_1) = n+1.$

From the definition of f, all the vertex and edge labels are at most n+1. The edge weights are
$wt(u_iv_i) = 2i+1, 1 \le i \le n;$
$wt(v_iv_{i+1}) = 2i+2, 1 \le i \le n;$
$wt(vv_i) = 2n+2+i, 1 \le i \le n.$

Hence the edge weights are $\{3,4,5,...,2n+1,2n+2,2n+3,...,3n+2\}$. Thus the vertex weights are

$$wt(u_i) = 1+i, 1 \le i \le n;$$

$$wt(v_i) = \begin{cases} 2n+5, & if \ i = 1 \\ n+3+2i, & if \ 2 \le i \le n-1 \\ 4n+3, & if \ i = n; \end{cases}$$

$$wt(v) = (n+1)^2.$$

Hence the vertex weights are $\{2,3,4,5,...,n+1,n+7,n+9,n+11,...,3n+1,2n+5,4n+3,(n+1)^2\}$ and all are distinct.

Case(ii): n is even.

$f(u_i) = f(v_i) = i, 1 \le i \le n;$
$f(u_iv_i) = 1, 1 \le i \le n;$
$f(v) = n+1;$
$f(vv_i) = n+1, 1 \le i \le n;$
$f(v_iv_{i+1}) = 1, 1 \le i \le n-1;$
$f(v_nv_1) = n+1.$

From the definition of f, all the vertex and edge labels are at most n+1. The edge weights are

$wt(u_iv_i) = 2i+1, 1 \le i \le n;$
$wt(v_iv_{i+1}) = 2i+2, 1 \le i \le n;$
$wt(vv_i) = 2n+2+i, 1 \le i \le n.$

Hence the edge weights are $\{3,4,5,...,2n+1,2n+2,2n+3,...,3n+2.\}$. Thus the vertex weights are

$$wt(u_i) = 1+i, 1 \le i \le n;$$

$$wt(v_i) = \begin{cases} 2n+5, & if\ i=1 \\ n+4+i, & if\ 2 \le i \le n-1 \\ 3n+4, & if\ i=n; \end{cases}$$

$$wt(v) = (n+1)^2.$$

Hence the vertex weights are $\{2, 3, 4, 5, ..., n+1, n+6, n+7, ..., 2n+3, 2n+5, 3n+4, (n+1)^2\}$ and all are distinct. From the above two cases, this labeling construction shows that $ts(H_n) \le n+1$. Combining this with the lower bound, we conclude that $ts(H_n) = n+1$. This completes the proof. Figure 1 shows a totally irregular total labeling of helm graph H_6.

□

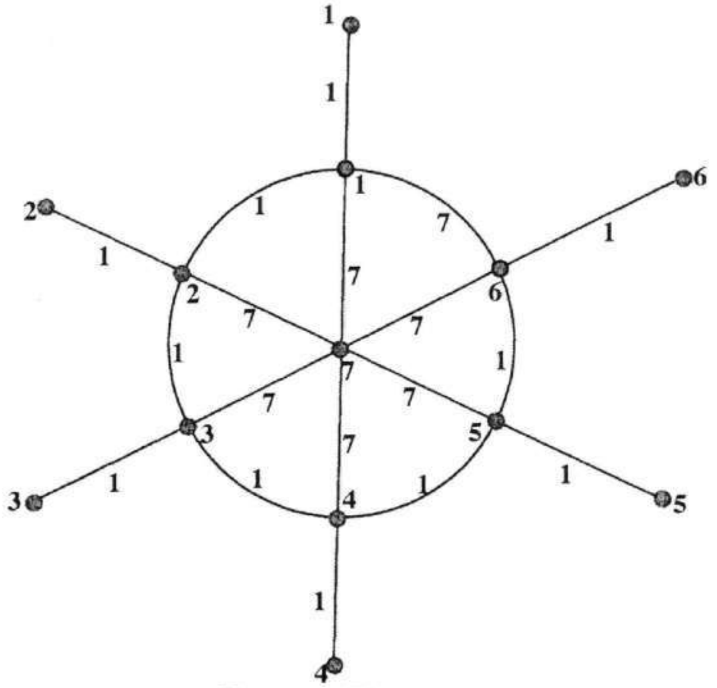

Figure: 1 $ts(H_6) = 7$

Theorem 2.2. *Let $n \ge 3$ and CH_n be a closed helm graph with 2n+1 vertices and 4n edges. Then $ts(CH_n) = \left\lceil \frac{4n+2}{3} \right\rceil, n \ge 3$.*

Proof. Since $|V(CH_n)| = 2n+1$ and $|E(CH_n)| = 4n$ by (2), (7) and (11) we have $ts(CH_n) \ge \left\lceil \frac{4n+2}{3} \right\rceil$. Let $k := \left\lceil \frac{4n+2}{3} \right\rceil$. For the reverse inequality, we define a total labeling f as follows:
$f : V \cup E \to \{1, 2, 3, \ldots, k\}$ by considering the following two cases.

Case(i): n is odd.

$f(u_i v_i) = 2n+2-k, 1 \le i \le n;$
$f(v_i) = i, 1 \le i \le n;$

$f(u_i) = k, 1 \leq i \leq n;$
$f(v) = 1;$
$f(vv_i) = 1, 1 \leq i \leq n;$
$f(u_iu_{i+1}) = k + 1 - i, 1 \leq i \leq n;$
$f(v_iv_{i+1}) = n + 1 - i, 1 \leq i \leq n - 1;$
$f(v_nv_1) = n + 1.$

From the definition of f, all the vertex and edge labels are at most k. The edge weights are

$wt(u_iv_i) = 2n + 2 + i, 1 \leq i \leq n;$
$wt(v_iv_{i+1}) = n + 2 + i, 1 \leq i \leq n;$
$wt(vv_i) = 2 + i, 1 \leq i \leq n;$
$wt(u_iu_{i+1}) = 3k + 1 - i.1 \leq i \leq n.$

Hence the edge weights are $\{3, 4, 5, ..., n + 2, n + 3, ..., 2n + 2, 2n + 3, ..., 3n + 2, 3k, 3k - 1, 3k - 2, 3k + 1 - n\}$.
Thus the vertex weights are

$$wt(u_i) = \begin{cases} 2k + n + 3, & if \ \ i = 1 \\ 2k + 2n + 5 - 2i, & if \ \ 2 \leq i \leq n; \end{cases}$$
$$wt(v_i) = \begin{cases} 4n + 6 - k - i, & if \ \ 1 \leq i \leq n - 1 \\ 4n + 6 - k, & if \ \ i = n; \end{cases}$$
$$wt(v) = (n + 1).$$

Hence the vertex weights are $\{n + 1, 4n + 5 - k, 4n + 4 - k, ..., 3n + 7 - k, 4n + 6 - k, 2k + n + 3, 2k + 2n + 1, 2k + 2n - 1, ..., 2k + 5\}$ and all are distinct.

Case(ii): n is even.

$f(u_i) = k, 1 \leq i \leq n;$
$f(v_i) = i, 1 \leq i \leq n;$
$f(u_iv_i) = 2n + 2 - k, 1 \leq i \leq n;$
$f(v) = 1;$
$f(vv_i) = 1, 1 \leq i \leq n;$
$f(v_iv_{i+1}) = n + 1 - i, 1 \leq i \leq n - 1;$
$f(v_nv_1) = n + 1;$
$$f(u_iu_{i+1}) = \begin{cases} k + 2 - 2i, & if \ \ 1 \leq i \leq \frac{n}{2} \\ k - 1 - 2n + 2i, & if \ \ \frac{n}{2} + 1 \leq i \leq n. \end{cases}$$
From the definition of f, all the vertex and edge labels are at most k. The edge weights are

$wt(u_iv_i) = 2n + 2 + i, 1 \leq i \leq n;$
$wt(v_iv_{i+1}) = n + 2 + i, 1 \leq i \leq n;$
$wt(vv_i) = 2 + i, 1 \leq i \leq n;$

$$wt(u_iu_{i+1}) = \begin{cases} 3k + 2 - 2i, & if \ 1 \leq i \leq \frac{n}{2} \\ 3k - 1 - 2n + 2i, & if \ \frac{n}{2} + 1 \leq i \leq n. \end{cases}$$

Hence the edge weights are $\{3, 4, 5, ..., n + 2, n + 3, n + 4, ..., 2n + 2, 2n + 3, ..., 3n + 2, 3k, 3k - 2, 3k - 4, ..., 3k + 1 - n, 3k + 2 - n, 3k + 5 - n, ..., 3k - 3, 3k - 1\}$.
Thus the vertex weights are

$wt(v) = (n + 1);$
$wt(v_i) = 4n + 6 - k - i, 1 \leq i \leq n;$

$$wt(u_i) = \begin{cases} 2k + 2n + 1, & if \ i = 1 \\ 2k + 2n + 8 - 4i, & if \ 2 \leq i \leq \frac{n}{2} \\ 2k + 5, & if \ i = \frac{n}{2} + 1 \\ 2k - 2 - 2n + 4i, & if \ \frac{n}{2} + 2 \leq i \leq n. \end{cases}$$

Hence the vertex weights are $\{n + 1, 4n + 5 - k, 4n + 4 - k, ..., 3n + 6 - k, 2k + 2n + 1, 2k + 2n, 2k + 2n - 4, 2k + 2n - 8, 2k + 5, 2k + 6, 2k + 10, 2k + 14, ..., 2k + 2n - 2\}$ and all are distinct. From the above two cases, this labeling construction shows that $ts(CH_n) \leq k$. Combining this with the lower bound, we conclude that $ts(CH_n) = k$. This completes the proof. Figure 2 shows a totally irregular total labeling of closed helm graph CH_6. □

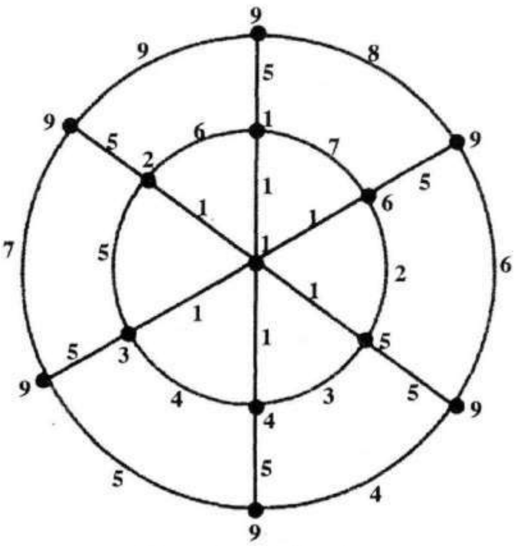

Figure 2: $ts(CH_6) = 9$.

Theorem 2.3. *Let $n \geq 3$ and DW_n be a double wheel graph with 2n+1 vertices and 4n edges. Then $ts(DW_n) = \left\lceil \frac{4n+2}{3} \right\rceil, n \geq 3$.*

Proof. Since $|V(DW_n)| = 2n+1$ and $|E(DW_n)| = 4n$ by (1), (7) and (11) we have $ts(DW_n) \geq \left\lceil \frac{4n+2}{3} \right\rceil$. Let $k = \left\lceil \frac{4n+2}{3} \right\rceil$. For the reverse inequality, we define a total labeling f as follows :

$f(v_i) = k, 1 \leq i \leq n;$
$f(u_i) = \left\lceil \frac{i+1}{2} \right\rceil, 1 \leq i \leq n-1;$
$f(u_n) = 1;$
$f(v) = \left\lceil \frac{k}{2} \right\rceil;$
$f(vv_i) = 2n+2-\left\lceil \frac{k}{2} \right\rceil - k + i, 1 \leq i \leq n;$

$$f(u_iu_{i+1}) = \begin{cases} 3 - \left\lceil \frac{i+1}{2} \right\rceil - \left\lceil \frac{i+2}{2} \right\rceil + i, & if\ 1 \leq i \leq n-2 \\ n+1-\left\lceil \frac{n}{2} \right\rceil, & if\ i = n-1 \\ 1, & if\ i = n; \end{cases}$$

$$f(vu_i) = \begin{cases} n+2-\left\lceil \frac{k}{2} \right\rceil - \left\lceil \frac{i+1}{2} \right\rceil + i, & if\ 1 \leq i \leq n-1 \\ 2n+1-\left\lceil \frac{k}{2} \right\rceil, & if\ i = n; \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 3n+2-2k+i, & if\ n \equiv 0\ (mod 3), 1 \leq i \leq n-1 \\ k, & if\ n \equiv 0\ (mod 3), i = n \\ 3n+2-2k+i, & if\ n \equiv 1,2\ (mod 3), 1 \leq i \leq n. \end{cases}$$

From the definition of f, all the vertex and edge labels are at most k. The edge weights are

$$wt(u_iu_{i+1}) = \begin{cases} 3+i, & if\ 1 \leq i \leq n-2 \\ n+2, & if\ i = n-1 \\ 3, & if\ i = n; \end{cases}$$

$wt(vu_i) = n+2+i, 1 \leq i \leq n;$
$wt(vv_i) = 2n+2+i, 1 \leq i \leq n;$

$$wt(v_iv_{i+1}) = \begin{cases} 3n+2+i, & if\ n \equiv 1,2\ (mod 3), 1 \leq i \leq n \\ 3n+2+i, & if\ n \equiv 0\ (mod 3), 1 \leq i \leq n-1 \\ 3k, & if\ n \equiv 0\ (mod 3), i = n. \end{cases}$$

Hence the edge weights are {3,4,5,...,n+2,n+3,n+4,...,2n+2,2n+3,...,3n+2 } and {3n+3,3n+4,...,4n+2} (or) {3n+3,3n+4,...,4n+1,3k }.Thus the vertex weights are

$$wt(u_i) = \begin{cases} 7+n-\left\lceil \frac{i+2}{2} \right\rceil - 2\left\lceil \frac{i+1}{2} \right\rceil - \left\lceil \frac{i}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil + 3i, & if\ 1 \leq i \leq n-2 \\ 4n+3-\left\lceil \frac{n-1}{2} \right\rceil - 2\left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil, & if\ i = n-1 \\ 3n+4-\left\lceil \frac{k}{2} \right\rceil - \left\lceil \frac{n}{2} \right\rceil, & if\ i = n; \end{cases}$$

$$wt(v_i) = \begin{cases} 9n-4k+8-\left\lceil \frac{k}{2} \right\rceil, & if\ n \equiv 1,2\ (mod3), i=1 \\ 8n-4k-\left\lceil \frac{k}{2} \right\rceil+5+3i, & if\ n \equiv 1,2\ (mod3), 2 \le i \le n \\ 5n-k+6-\left\lceil \frac{k}{2} \right\rceil, & if\ n \equiv 0\ (mod3), i=1. \\ 8n-4k-\left\lceil \frac{k}{2} \right\rceil+5+3i, & if\ n \equiv 0\ (mod3), 2 \le i \le n-1 \\ 7n+3-\left\lceil \frac{k}{2} \right\rceil-k, & if\ n \equiv 0\ (mod3), i=n; \end{cases}$$

$$wt(v) = \left\lceil \frac{k}{2} \right\rceil (1-2n)+5n-nk+4n^2-\sum_{i=1}^{n-1} \left\lceil \frac{i+1}{2} \right\rceil - 1.$$

Hence the vertex weights are { $n+5-\left\lceil \frac{k}{2} \right\rceil, \dots, 4n+1-2\left\lceil \frac{n}{2} \right\rceil - 2\left\lceil \frac{n+1}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil, 4n+3-\left\lceil \frac{n-1}{2} \right\rceil - 2\left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil, 3n+4-\left\lceil \frac{k}{2} \right\rceil - \left\lceil \frac{n}{2} \right\rceil$} and {$9n-4k+8-\left\lceil \frac{k}{2} \right\rceil, 8n-4k+11-\left\lceil \frac{k}{2} \right\rceil, 8n-4k+14-\left\lceil \frac{k}{2} \right\rceil, \dots, 11n-4k+5-\left\lceil \frac{k}{2} \right\rceil$}(or){$5n-k+6-\left\lceil \frac{k}{2} \right\rceil, 7n+3-k-\left\lceil \frac{k}{2} \right\rceil, 8n-4k+11-\left\lceil \frac{k}{2} \right\rceil, 8n-4k+14-\left\lceil \frac{k}{2} \right\rceil, \dots, 11n-4k+2-\left\lceil \frac{k}{2} \right\rceil$} all are distinct. This labeling construction shows that $ts(DW_n) \le k$. Combining this with the lower bound, we conclude that $ts(DW_n) = k$. This completes the proof. Figure 3 shows a totally irregular total labeling of double wheel graph DW_6. □

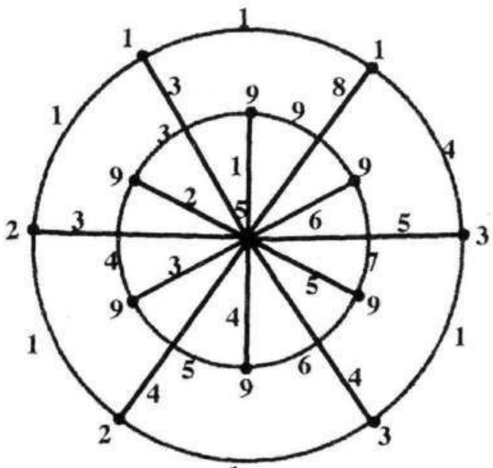

Figure 3: $tvs(DW_6) = 9$.

Theorem 2.4. *Let $n \ge 3$ and Fl_n be a flower graph with $2n+1$ vertices and $4n$ edges. Then $ts(Fl_n) = \left\lceil \frac{4n+2}{3} \right\rceil, n \ge 3$.*

Proof. Since $|V(Fl_n)| = 2n+1$ and $|E(Fl_n)| = 4n$ by (3), (10) and (11) we have $ts(Fl_n) \ge \left\lceil \frac{4n+2}{3} \right\rceil$. Let $k = \left\lceil \frac{4n+2}{3} \right\rceil$. For the reverse inequality, we define a total labeling f by considering the following two cases.

Case(i): $3 \le n \le 6$ and $n = 8$.

For Fl_3

$f(v) = 1, f(v_1) = 3, f(v_2) = 4, f(v_3) = 5, f(u_1) = f(u_2) = f(u_3) = 1, f(v_1v_2) = f(v_2v_3) = f(v_3v_1) = 5, f(vv_1) = f(vv_2) = f(vv_3) = 2, f(v_1u_1) = f(v_2u_2) =$

$f(v_3u_3) = 5, f(vu_1) = f(vu_2) = f(vu_3) = 3.$

For Fl_4

$f(v) = 1, f(v_1) = 4, f(v_2) = 5, f(v_3) = 6, f(v_4) = 6, f(u_1) = f(u_2) = f(u_3) = 1, f(u_4) = 2, f(v_1v_2) = f(v_2v_3) = f(v_3v_4) = f(v_4v_1) = 6, f(vv_1) = f(vv_2) = f(vv_3) = 2, f(vv_4) = 3, f(v_1u_1) = f(v_2u_2) = f(v_3u_3) = f(v_4u_4) = 6, f(vu_1) = 1, f(vu_2) = 2, f(vu_3) = 3, f(vu_4) = 3.$

For Fl_5

$f(v) = 1, f(v_1) = 5, f(v_2) = 6, f(v_3) = 7, f(v_4) = 7, f(v_5) = 7, f(u_1) = f(u_2) = f(u_3) = 1, f(u_4) = 2, f(u_5) = 3, f(v_1v_2) = 7, f(v_2v_3) = 6, f(v_3v_4) = 8, f(v_4v_5) = 7, f(v_5v_1) = 8, f(vv_1) = f(vv_2) = f(vv_3) = 2, f(vv_4) = 3, f(vv_5) = 4, f(v_1u_1) = f(v_2u_2) = f(v_3u_3) = f(v_4u_4) = f(v_5u_5) = 7, f(vu_1) = f(vu_2) = f(vu_3) = 1, f(vu_4) = 2, f(vu_5) = 3.$

For Fl_6

$f(v) = 1, f(v_1) = 6, f(v_2) = 7, f(v_3) = 8, f(v_4) = 8, f(v_5) = f(v_6) = 9, f(u_1) = f(u_2) = f(u_3) = f(u_5) = 1, f(u_4) = f(u_6) = 2, f(v_1v_2) = 8, f(v_2v_3) = 7, f(v_3v_4) = f(v_4v_5) = f(v_5v_6) = f(v_6v_1) = 9, f(vv_1) = f(vv_2) = f(vv_3) = 2, f(vv_4) = f(vv_5) = 3, f(vv_6) = 4, f(v_1u_1) = f(v_2u_2) = f(v_3u_3) = f(v_4u_4) = 8, f(v_5u_5) = f(v_6u_6) = 9, f(vu_1) = 1, f(vu_2) = 2, f(vu_3) = 3, f(vu_4) = 3, f(vu_5) = f(vu_6) = 5.$

For Fl_8

$f(v) = 1, f(v_1) = 8, f(v_2) = 9, f(v_3) = 10, f(v_4) = f(v_5) = f(v_6) = f(v_7) = f(v_8) = 11, f(u_1) = f(u_2) = f(u_3) = f(u_4) = 1, f(u_5) = 2, f(u_6) = 3, f(u_7) = 4, f(u_8) = 5, f(v_1v_2) = 10, f(v_2v_3) = 9, f(v_3v_4) = 8, f(v_4v_5) = 12, f(v_5v_6) = 11, f(v_6v_7) = 10, f(v_7v_8)$
$= 9, f(v_8v_1) = 11, f(vv_1) = f(vv_2) = f(vv_3) = f(vv_4) = 2, f(vv_5) = 3, f(vv_6) = 4, f(vv_7) = 5, f(vv_8) = 6, f(v_1u_1) = f(v_2u_2) = f(v_3u_3) = f(v_4u_4) = f(v_5u_5) = f(v_6u_6) = f(v_7u_7) = f(v_8u_8) = 10, f(vu_1) = 1, f(vu_2) = 2, f(vu_3) = 3, f(vu_4) = f(vu_5) = f(vu_6) = f(vu_7) = f(vu_8) = 4.$

Case(ii): $n \geq 7, n \neq 8.$

We construct the function f as follows:

$f(v) = 1;$

$$f(v_i) = \begin{cases} n - 1 + i, & if\ 1 \leq i \leq k - n + 1 \\ k, & if\ k - n + 2 \leq i \leq n; \end{cases}$$

$$f(u_i) = \begin{cases} 1, & if\ 1 \le i \le k-n+1 \\ i-k+n, & if\ k-n+2 \le i \le n-1 \\ k-n+1 & if\ i=n; \end{cases}$$

$$f(vu_i) = \begin{cases} i, & if\ 1 \le i \le k-n+1 \\ k-n+1, & if\ k-n+2 \le i \le n-1; \\ 2n-k & if\ i=n; \end{cases}$$

$$f(vv_i) = \begin{cases} 2, & if\ 1 \le i \le k-n+1 \\ n-k+1+i, & if\ k-n+2 \le i \le n; \end{cases}$$

$$f(v_iu_i) = \begin{cases} n+2, & if\ 1 \le i \le n-1 \\ 4n+1-2k, & if\ i=n; \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} n+3-i, & if\ 1 \le i \le k-n \\ n+3, & if\ i=n; \end{cases}$$

$$f(v_iv_{i-1}) = 3n+4-k-i, k-n+2 \le i \le n.$$

From the definition of f, all the vertex and edge labels are at most k. The edge weights are

$$wt(vu_i) = 2+i, 1 \le i \le n;$$
$$wt(vv_i) = n+2+i, 1 \le i \le n;$$
$$wt(v_iu_i) = \begin{cases} 2n+2+i, & if\ 1 \le i \le n-1 \\ 3n+2, & if\ i=n \end{cases}$$
$$wt(v_iv_{i+1}) = \begin{cases} 3n+2+i, & if\ 1 \le i \le k-n \\ 2n+k+3, & if\ i=n; \end{cases}$$
$$wt(v_iv_{i-1}) = 3n+4+k-i, k-n+2 \le i \le n.$$

Hence the edge weights are $\{3,4,5,...,n+2,n+3,n+4,...,2n+2,2n+3,...,3n+2,3n+3,...,2n+2+k,2n+3+k,4n+2,4n+1,4n,2n+4+k\}$. Thus the vertex weights are

$$wt(u_i) = \begin{cases} n+3+i, & if\ 1 \le i \le n-1 \\ 5n+2-2k, & if\ i=n. \end{cases}$$

$$wt(v_i) = \begin{cases} 4n+10-i, & if\ 1 \le i \le k-n \\ 7n+9-2k, & if\ i=k-n+1 \\ 8n+10-2k-i, & if\ k-n+2 \le i \le n-1; \\ 9n+9-3k, & if\ i=n; \end{cases}$$

$$wt(v) = k + 3 + \sum_{i=k-n+2}^{n} (n - k + 1 + i) + \sum_{i=1}^{k-n+1} (i) + \sum_{i=k-n+2}^{n-1} (k - n + 1).$$

Hence the vertex weights are $\{n + 4, n + 5, ..., 2n + 2, 5n + 2 - 2k, 4n + 9, 4n + 8, ..., 5n + 10 - k, 7n + 9 - 2k, 9n - 3k + 8, 9n - 3k + 7, ..., 7n - 2k + 11, 9n + 9 - 3k\}$ and all are distinct. From the above two cases, this labeling construction shows that $ts(Fl_n) \leq k$. Combining this with the lower bound, we conclude that $ts(Fl_n) = k$. This completes the proof. Figure 4 shows a totally irregular total labeling of flower graph Fl_6. □

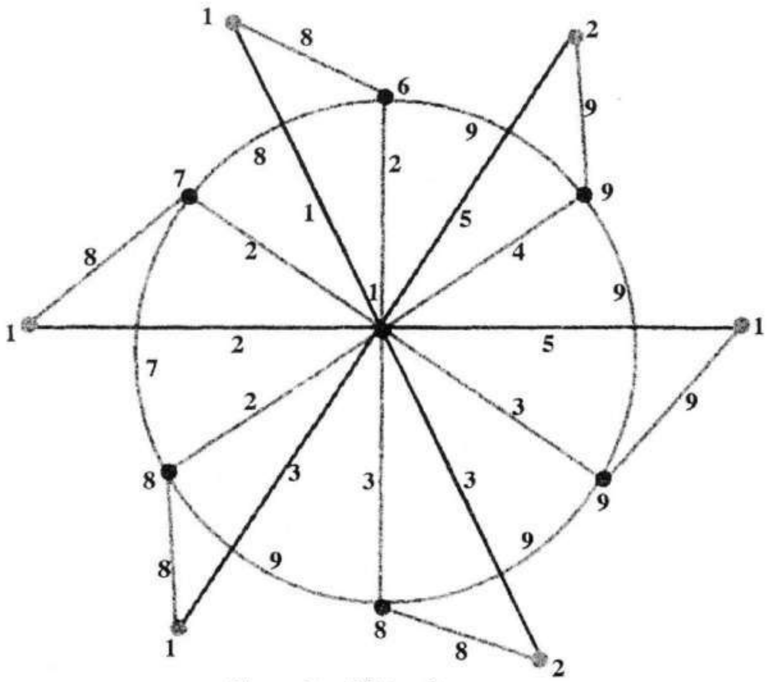

Figure 4: $ts(Fl_6) = 9$

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